Arc-disjoint in-trees in directed graphs

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Arc-disjoint In-trees in Directed Graphs

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Abstract

Given a directed graph $D = (V, A)$ and a set of specified vertices $S = \{s_1, \ldots, s_d\} \subseteq V$ with $|S| = d$ and a function $f : S \to \mathbb{N}$ where $\mathbb{N}$ denotes the set of natural numbers, we present a necessary and sufficient condition that there exist $\sum_{s_i \in S} f(s_i)$ arc-disjoint in-trees denoted by $T_{i,1}, T_{i,2}, \ldots, T_{i,f(s_i)}$ for every $i = 1, \ldots, d$ such that $T_{i,1}, \ldots, T_{i,f(s_i)}$ are rooted at s_i and each $T_{i,j}$ spans vertices from which s_i is reachable. This generalizes the result of Edmonds [2], i.e., the necessary and sufficient condition that for a directed graph $D = (V, A)$ with a specified vertex $s \in V$, there are k arc-disjoint in-trees rooted at s each of which spans V. Furthermore, we extend another characterization of packing in-trees of Edmonds [1] to the one in our case.

1 Introduction

Let $D = (V, A)$ be a directed graph which may have parallel arcs. A vertex v is said to be *reachable* from a vertex u when there is a path from u to v. We denote by $e = uv$ an arc e whose tail and head are u and v, respectively. If $e = uv$ has no parallel arc, we may simply write uv. For $X, Y \subseteq V$, let $\delta(X, Y; D) = \{e = uv \in A : u \in X, v \in Y\}$. For $W \subseteq V$, we write $\delta^+(W; D)$ and $\delta^-(W; D)$ instead of $\delta(W, V \setminus W; D)$ and $\delta(V \setminus W, W; D)$, respectively. For $W \subseteq V$, let $D[W]$ be a subgraph of D induced by W. For $u, v \in V$, we denote by $\lambda(u, v; D)$ the local arc connectivity from u to v in D, i.e.,

$$\begin{aligned}(1.1)\quad \lambda(u, v; D) &= \min\{|\delta^-(W; D)| : u \notin W, v \in W, W \subseteq V\}.\end{aligned}$$

We can see from (1.1) that for every $u, v \in V$, $W \subseteq V$ with $u \notin W$ and $v \in W$.

$$(1.2)\qquad \lambda(u, v; D) \leq |\delta^-(W; D)|.$$

Notice that $\lambda(u, v; D)$ is equal to the maximum number of arc-disjoint paths from u to v in D by Menger's Theorem (see Corollary 9.1b in Chapter 9 of [6]).

Background: In 1973, Edmonds gave a constructive proof of the following theorem.

THEOREM 1.1. ([2]) *Given a directed graph $D = (V, A)$ with a specified vertex $s \in V$, there exist k arc-disjoint in-trees rooted at s each of which spans V if and only if $\lambda(v, s; D) \geq k$ holds for every $v \in V \setminus \{s\}$.*

Alternative proofs are found in [5, 7]. In this paper, we generalize this theorem as follows. We are given a set of specified vertices $S = \{s_1, \ldots, s_d\} \subseteq V$ with $|S| = d$ and a function $f : S \to \mathbb{N}$ where $\mathbb{N}$ denotes the set of natural numbers, and we will present a necessary and sufficient condition that there exist $\sum_{s_i \in S} f(s_i)$ arc-disjoint in-trees denoted by $T_{i,1}, T_{i,2}, \ldots, T_{i,f(s_i)}$ for every $i = 1, \ldots, d$ that $T_{i,1}, \ldots, T_{i,f(s_i)}$ are rooted at s_i and each $T_{i,j}$ spans vertices from which s_i is reachable. As shown below, in the previous papers such as [6] which considered the generalization of Theorem 1.1 to the case that allows D to have multiple specified vertices, they assumed that every vertex $s_i \in S$ is reachable from every vertex $v \in V$, while in this paper we do not.

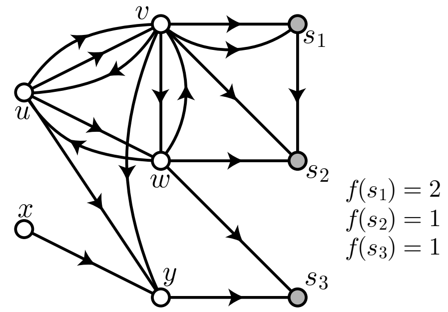

Figure 1: Directed graph D and function f.

For example, given a directed graph D in Figure 1 with $S = \{s_1, s_2, s_3\}$ and $f(s_1) = 2, f(s_2) = 1, f(s_3) = 1$, the set of vertices from which s_1 is reachable is equal to $\{u, v, w, s_1\}$, and the set of vertices from which s_2 is reachable is equal to $\{u, v, w, s_1, s_2\}$, and the set of vertices from which s_3 is reachable is equal to

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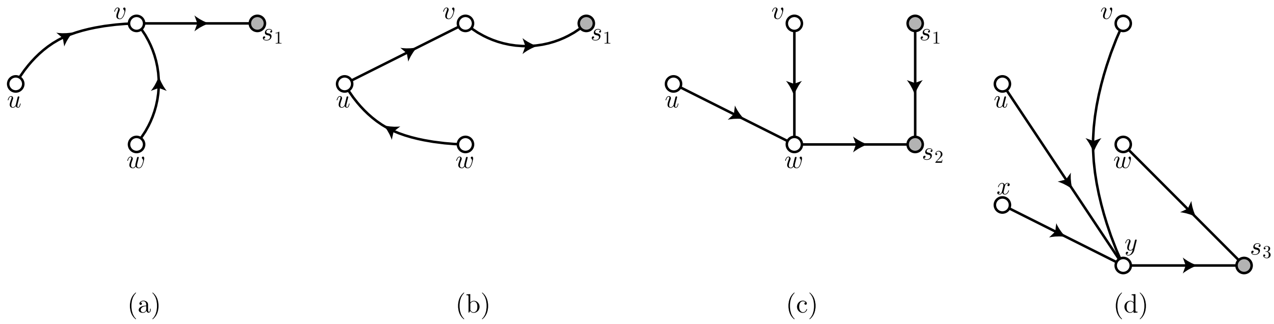

Figure 2: (a) $T_{1,1}$. (b) $T_{1,2}$. (c) $T_{2,1}$. (d) $T_{3,1}$.

$\{u, v, w, x, y, s_3\}$. We see that $T_{1,1}$, $T_{1,2}$, $T_{2,1}$, and $T_{3,1}$ shown in Figure 2 are arc-disjoint, and span vertices from which s_1, s_2 and s_3 are reachable, respectively.

Main result: Here we give the precise description of the main theorem in this paper. We first introduce necessary notations. For each $v \in V$, $R(v)$ denotes the set of vertices in S which are reachable from v. For $i = 1, \dots, d$, V_i denotes the set of vertices in V from which s_i is reachable. $D^* = (V^*, A^*)$ is a directed graph obtained from D by adding vertex s^* and connecting s_i to s^* with $f(s_i)$ parallel arcs (see Figure 3).

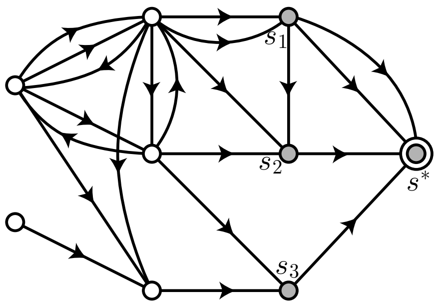

Figure 3: Transformed graph D^*.

Moreover, we define $f(S') = \sum_{s_i \in S'} f(s_i)$ for each $S' \subseteq S$. Then, the main theorem which we will prove in this paper is described as follows.

THEOREM 1.2. *Given a directed graph $D = (V, A)$ with a set of specified vertices $S = \{s_1, \dots, s_d\}$ and a function $f\colon S \to \mathbb{N}$, there exist $f(S)$ arc-disjoint in-trees denoted by $T_{i,1}, \dots, T_{i,f(s_i)}$ for every $i = 1, \dots, d$ such that $T_{i,1}, \dots, T_{i,f(s_i)}$ are rooted at s_i and each $T_{i,j}$ spans V_i if and only if $\lambda(v, s^*; D^*) \geq f(R(v))$ holds for every $v \in V$.*

Theorem 1.1 is a special case of Theorem 1.2 with $S = \{s\}$ and $f(s) = k$. In this case, from the definition of D^*, it is easy to see that $\lambda(v, s; D) \geq k$ holds for every $v \in V \setminus \{s\}$ if and only if $\lambda(v, s^*; D^*) \geq f(s)$ holds for every $v \in V$.

Non-triviality: Theorem 1.2 of the case where $R(v) = S$ holds for every $v \in V$ is known (see Corollary 53.1a in Chapter 53 of [6]).

THEOREM 1.3. ([6]) *There exist $f(S)$ arc-disjoint in-trees denoted by $T'_{i,1}, \dots, T'_{i,f(s_i)}$ for every $i = 1, \dots, d$ such that $T'_{i,1}, \dots, T'_{i,f(s_i)}$ are rooted at s_i and each $T'_{i,j}$ spans V if and only if $\lambda(v, s^*; D^*) \geq f(S)$ holds for every $v \in V$.*

It apparently seems that Theorem 1.2 can be directly derived from Theorem 1.3 by transforming a directed graph $D = (V, A)$ by adding $f(s_i)$ arcs from every vertex not in V_i to s_i. But this is not the case. To see this, let us consider a directed graph $D = (V, A)$ in Figure 4(a) with $S = \{s_1, s_2\}$ and $f(s_1) = 1$, $f(s_2) = 1$, where $V_1 = \{u, v, w, s_1\}$ and $V_2 = \{u, v, x, s_2\}$ hold. Now we add arcs xs_1, s_2s_1, ws_2, and s_1s_2 to A so that $R(v) = S$ holds for every $v \in V$ (Figure 4(b)). Let D' be the resulting graph.

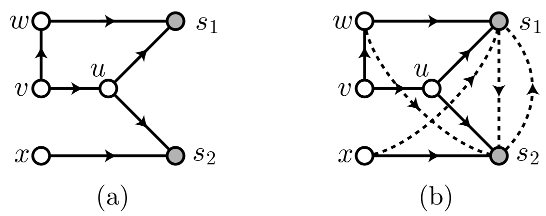

Figure 4: (a) Input directed graph D. (b) Transformed graph D'.

From Theorem 1.3, there exist two arc-disjoint in-trees in D' denoted by $T'_{1,1}$ and $T'_{2,1}$ such that $T'_{1,1}$ and $T'_{2,1}$ span V, and are rooted at s_1 and s_2, respectively. However, removing arcs that are added to obtain D' from $T'_{1,1}$ and $T'_{2,1}$ does not always produce the desired $T_{1,1}$ and $T_{2,1}$ such that $T_{1,1}$ is rooted at s_1 and spans V_1, and $T_{2,1}$ is rooted at s_2 and spans V_2. For $T'_{1,1}$ and $T'_{2,1}$ which are respectively illustrated in the left side

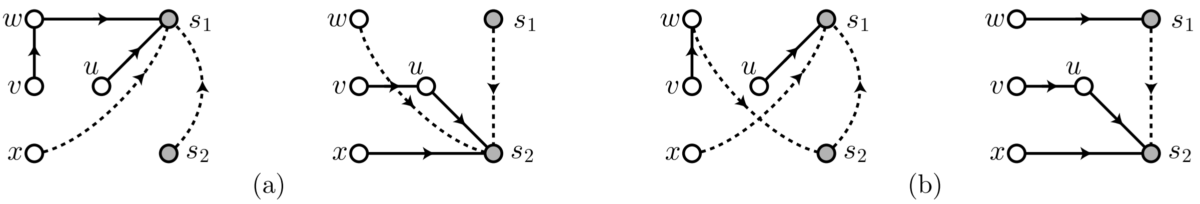

Figure 5: (a) Arc disjoint in-trees $T'_{1,1}$ and $T'_{2,1}$ for which removing arcs added (dotted arcs) results in $T_{1,1}$ and $T_{2,1}$ that satisfy the statement of Theorem 1.2. (b) Arc disjoint in-trees $T'_{1,1}$ and $T'_{2,1}$ for which removing arcs added results in $T_{1,1}$ and $T_{2,1}$ that do not satisfy the statement of Theorem 1.2.

and the right side of Figure 5(a), $T_{1,1}$ and $T_{2,1}$ obtained from $T'_{1,1}$ and $T'_{2,1}$ by simply removing arcs added to D (dotted arcs) satisfy the statement of Theorem 1.2. However, it is not the case as is seen from Figure 5(b) for $T'_{1,1}$ and $T'_{2,1}$ which are respectively illustrated in the left side and the right side of Figure 5(b). Therefore, we can see that Theorem 1.2 can not be immediately derived from Theorem 1.3.

Motivation: In our recent paper [4], we considered the *evacuation problem* defined on *dynamic network* and showed that this problem can be efficiently solved if the following property holds for the underlying directed graph $D^\circ = (V^\circ, A^\circ)$ and a sink $s^\circ \in V^\circ$ of a given dynamic network: For $P = \{s_1, \dots, s_d\}$ which is a set of vertices in V° incident to s°, there exists $|\delta^-(\{s^\circ\}; D^\circ)|$ arc-disjoint in-trees denoted by $T_{i,j}, \dots, T_{i,|\delta(\{s_i\},\{s^\circ\};D^\circ)|}$ for every $i = 1, \dots, d$ such that $T_{i,j}, \dots, T_{i,|\delta(\{s_i\},\{s^\circ\};D^\circ)|}$ are rooted at s_i and each $T_{i,j}$ spans from vertices s_i is reachable. This property is the same as Theorem 1.2 by setting $D = D^\circ \setminus \{s^\circ\}$, $S = P$, and $f(s_i) = |\delta(\{s_i\}, \{s^\circ\}; D^\circ)|$ for $s_i \in P$ where $D^\circ \setminus \{s^\circ\}$ denotes the directed graph obtained by removing s° and arcs incident to s° from D°. In [4], we proved Theorem 1.2 only for the case where D is *acyclic*. In this paper, we extend the result in [4] to the case where D° is allowed to have cycles.

Organization: Section 2 gives the proof of Theorem 1.2. In Section 3, we extend another characterization of packing in-trees of Edmonds [1] to the one in our case by using Theorem 1.2.

2 Proof of Theorem 1.2

It is not difficult to see that "only if-part" holds. We then prove the "if-part". That is, we assume that

$$\lambda(v, s^*; D^*) \geq f(R(v)) \text{ for every } v \in V. \tag{2.3}$$

We prove the theorem by induction on $f(S)$. In the case of $f(S) = 1$, the theorem clearly holds from $|S| = 1$.

We consider the case of $f(S) > 1$. Let us fix $i \in \{1, \dots, d\}$ and $e_i \in \delta(\{s_i\}, \{s^*\}; D^*)$. To prove the theorem by induction on $f(S)$, we will find an in-tree in D^* denoted by $T = (W, B)$ with $W \subseteq V_i \cup \{s^*\}$ such that T is rooted at s^* and satisfies (F0) and (F1).

(F0) $\delta^-(\{s^*\}; T) = \{e_i\}$, i.e., T has only one arc e_i incident to s^*.

(F1) For every $v \in V$,

$$\lambda(v, s^*; D^* \setminus B) \geq \begin{cases} f(R(v)) - 1, & \text{if } v \in V_i, \\ f(R(v)), & \text{if } v \in V \setminus V_i. \end{cases}$$

where $D \setminus A'$ denotes the directed graph obtained by removing A' from D, i.e., $D \setminus A' = (V, A \setminus A')$ for each $A' \subseteq A$.

If we can find an in-tree T rooted at s^* which spans V_i and satisfies (F0) and (F1), $T[V_i]$ is an in-tree rooted at s_i since a path from every $v \in V_i$ to s^* in T contains s_i from (F0). Moreover, since T does not contain any arc $s_j s^*$ for $j \neq i$ from (F0),

$$|\delta(\{s_j\}, \{s^*\}; D^* \setminus B)| = \begin{cases} f(s_j) - 1, & \text{if } j = i, \\ f(s_j), & \text{if } j \neq i. \end{cases}$$

Hence we can regard $D^* \setminus B$ as D^* for the case of $f(S) - 1$, and the proof is done by induction.

Here we remark that every in-tree rooted at s^* which spans V_i dose not always satisfy (F0) and (F1). For example, an in-tree in Figure 6(b) satisfies (F0) and (F1) in a directed graph D^* in Figure 6(a) and spans V_1, while an in-tree in Figure 6(c) denoted by $T = (W, B)$ does not satisfy (F1) since $\lambda(v, s^*; D^* \setminus B) = 0$ holds.

We call an in-tree $T = (W, B)$ with $W \subseteq V_i \cup \{s^*\}$ *feasible* if T is rooted at s^* and satisfies (F0) and (F1). For a feasible in-tree $T = (W, B)$, we call an arc $e = xy$ *eligible* when e satisfies

(E0) $x \in V_i \setminus W$ and $y \in W$,

(E1) $T' = (W \cup \{x\}, B \cup \{e\})$ is feasible.

That is, if there exists an eligible arc e for a feasible in-tree T, we can extend T by adding e while maintaining the feasibility of the augmented in-tree.

Framework of Proof: We will prove the existence

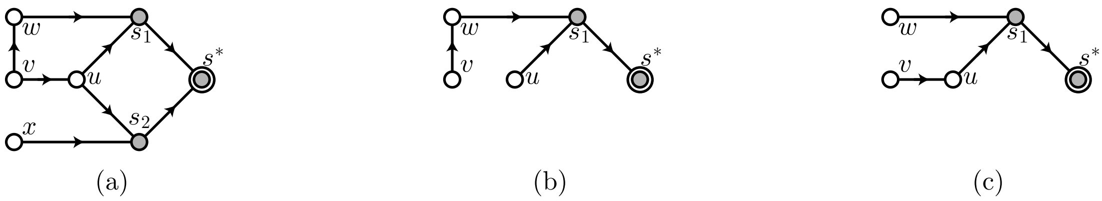

Figure 6: (a) D^* with $S = \{s_1, s_2\}$ and $f(s_1) = 1, f(s_2) = 1$. (b) Feasible in-tree. (c) Infeasible in-tree.

of a feasible in-tree T which spans V_i by induction on the number of vertices of T. First we prove Lemma 2.1 which says for the basis of induction that $T = (\{s^*, s_i\}, \{e_i\})$ is feasible. Then we prove that for any feasible in-tree which does not span V_i there exists an eligible arc. For this, we introduce the notion of *critical set* which says that any arc entering the critical set is not eligible. After this, we prove by using Lemmas 2.4 and 2.5 that there always exists an eligible arc for any feasible in-tree which does not span V_i. Lemma 2.3 which is the main contribution of this paper is used in the proofs of Lemmas 2.4 and 2.5.

Novelty: Our proof that we can construct a feasible in-tree that spans V_i is based on the proof of Theorem 1.1 of Lovász [5]. However, recall that in Theorem 1.1, the local arc connectivity from every $v \in V \setminus \{s\}$ to s is assumed to be at least a *constant* k which does not depend on v. Thus, given an in-tree $T = (W, B)$ rooted at s such that $\lambda(v, s; D \setminus B) \geq k - 1$ holds and T does not span V, we can determine whether an arc e can be added to T while maintaining $\lambda(v, s; D \setminus (B \cup \{e\})) \geq k - 1$ for every $v \in V \setminus \{s\}$ by simply testing whether $|\delta^-(V'; D \setminus (B \cup \{e\}))|$ is at least $k-1$ for every $V' \subsetneq V$ with $s \in V'$. But in our case, the condition of the local arc connectivity from each $v \in V$ to s^* in D^* is not uniform. Hence, given a feasible in-tree $T = (W, B)$ which does not span V_i, to determine whether an arc e is eligible, we have to test whether $|\delta^-(V'; D^* \setminus (B \cup \{e\}))|$ is at least $\min\{f(R(v)) - 1 \colon v \in V_i \setminus V'\}$ and $\min\{f(R(v)) \colon v \in V \setminus (V_i \cup V')\}$ for every $V' \subsetneq V^*$ with $s^* \in V'$. This makes the proof of Theorem 1.2 much harder. To cope with this hardness, we will introduce Lemma 2.3. The proof of Lemma 2.3 is trivial for the case of Theorem 1.1 and Theorem 1.3, i.e, the case where S is a singleton and every vertex in S is reachable from every $v \in V$, respectively. However the proof of Lemma 2.3 for the case of Theorem 1.2 is not trivial. The proof of Lemma 2.3 is the main contribution of this paper.

2.1 Construction of feasible in-tree We first prove the following lemma.

LEMMA 2.1. *For every subset A' of arcs in $D^*[V_i \cup \{s^*\}]$ and $v \in V \setminus V_i$, $\lambda(v, s^*; D^* \setminus A') = \lambda(v, s^*; D^*)$ holds. That is, the local arc connectivity from v to s^* does not change by removing arcs in A' from D^*.*

Proof. In D^*, any path from $v \in V \setminus V_i$ to s^* does not pass through any vertex in V_i from the definition of V_i. Thus, removing arcs of $D^*[V_i \cup \{s^*\}]$ does not reduce the local arc connectivity from $v \in V \setminus V_i$ to s^*. $\square$

Now we will prove that there exists a feasible in-tree $T = (W, B)$ which spans V_i by induction on $|W|$. For the basis of induction, it holds that $T = (\{s^*, s_i\}, \{e_i\})$ is feasible from Lemma 2.1.

Suppose that we have a feasible in-tree $T = (W, B)$ which does not span V_i. Then, we will prove that there always exists an eligible arc for T. Since T has to satisfy (F0), an arc whose head is s^* is not eligible. Furthermore, since $\delta(V \setminus V_i, V_i; D^*) = \emptyset$ follows from the definition of V_i and $W \setminus \{s^*\} \subseteq V_i$ holds,

(2.4) $x \in V_i \backslash W$ for any $e = xy \in \delta^-(W \backslash \{s^*\}; D^* \backslash B)$,

i.e., every $e \in \delta^-(W \setminus \{s^*\}; D^* \setminus B)$ satisfies (E0). Thus, to prove that there exists an eligible arc, it is sufficient to prove that there exists an arc $e = xy \in \delta^-(W \setminus \{s^*\}; D^* \setminus B)$ such that $T' = (W \cup \{x\}, B \cup \{e\})$ satisfies (F1).

It is obvious that $\delta^-(W \setminus \{s^*\}; D^* \setminus B) \neq \emptyset$ since T does not span V_i and s_i is reachable from every $v \in V_i \setminus W$ in D. However, not every arc in $\delta^-(W \setminus \{s^*\}; D^* \setminus B)$ is eligible. Consider the case where there exists $v \in V_i$ with $\lambda(v, s^*; D^* \setminus B) = f(R(v)) - 1$. In this case, from (1.1), there must exist $X \subseteq V^*$ with $s^* \in X$, $v \notin X$, and $|\delta^-(X; D^* \setminus B)| = f(R(v)) - 1$, i.e., $\delta^-(X; D^* \setminus B)$ is the minimum v-s^* cut in $D^* \setminus B$. Then, an arc $e = xy \in \delta^-(W \setminus \{s^*\}; D^* \setminus B)$ such that $e \in \delta^-(X; D^* \setminus B)$ is not eligible since $T' = (W \cup \{x\}, B \cup \{e\})$ violates (F1) for v.

For example, assume that for D^* in Figure 6(a), we currently have a feasible in-tree $T = (W, B)$ such that $W = \{s^*, s_1, u, w\}$ and $B = \{s_1 s^*, w s_1, u s_1\}$. Figure 7(a) shows $D^* \setminus B$. Suppose we add vu to T and let $T' = (W', B')$ be the resulting in-tree. Then, $\lambda(v, s^*; D^* \setminus B') = 0$, and hence T' does not satisfy (F1). That is, vu is not eligible. In this case, letting

$X = \{s^*, s_1, s_2, u, x\}$, $|\delta^-(X; D^* \setminus B)| = 1 = f(R(v)) - 1$ and $vu \in \delta^-(X; D^* \setminus B)$ holds.

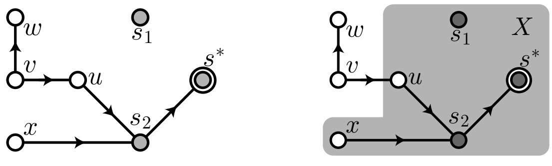

Figure 7: (a) $D^* \setminus B$. (b) $X = \{s^*, s_1, s_2, u, x\}$.

Here we give the precise description of the above discussion. A vertex set $X \subseteq V^*$ with $s^* \in X$ is called *critical* when X satisfies the following conditions.

(C0) $V_i \setminus (X \cup W) \neq \emptyset$.

(C1) $|\delta^-(X; D^* \setminus B)| = f(R(v)) - 1$ for some $v \in V_i \setminus X$.

Lemma 2.2. *Every $e = xy \in \delta^-(W \setminus \{s^*\}; D^* \setminus B)$ is eligible if there exists no critical set $X \subseteq V^*$ with $e \in \delta^-(X; D^* \setminus B)$.*

Proof. It is sufficient to prove that $T' = (W \cup \{x\}, B \cup \{e\})$ satisfies (F1). Suppose that for an arc $e = xy$ that satisfies the lemma assumption, T' does not satisfy (F1). Since from Lemma 2.1 the local arc connectivity from every $w \in V \setminus V_i$ to s^* does not change by removing arc in $D[V_i]$ (notice that e is an arc in $D[V_i]$ from (2.4)), there exists $v \in V_i$ such that $\lambda(v, s^*; D^* \setminus (B \cup \{e\})) \leq f(R(v)) - 2$. From (1.1), there exists $Y \subseteq V^*$ with $s^* \in Y$ and $v \notin Y$ such that

(2.5) $$|\delta^-(Y; D^* \setminus (B \cup \{e\}))| \leq f(R(v)) - 2.$$

We will show that Y satisfies (C0) and (C1), and $e \in \delta^-(Y; D^* \setminus B)$ holds, which contradicts that e satisfies the lemma assumption.

Since T satisfies (F1), $|\delta^-(Y; D^* \setminus B)| \geq f(R(v)) - 1$ follows from (1.2). Thus, since $|\delta^-(Y; D^* \setminus B)| - |\delta^-(Y; D^* \setminus (B \cup \{e\}))|$ is at most one, $|\delta^-(Y; D^* \setminus B)|$ must be equal to $f(R(v)) - 1$ (i.e., Y satisfies (C1)) and $e \in \delta^-(Y; D^* \setminus B)$ holds from (2.5).

Since $x \in V_i \setminus W$ follows from (2.4) and $x \notin Y$ follows from $e \in \delta^-(Y; D^* \setminus B)$, $x \in V_i \setminus (Y \cup W)$ holds. Thus, Y satisfies (C0). This completes the proof. □

We now consider the case where there exists a critical set. From now on, we prove that in this case, there always exists an eligible arc $e \in \delta^-(W \setminus \{s^*\}; D^* \setminus B)$. To prove this, let us fix $X_{\max}$ as a critical set which satisfies

(2.6) $$|X_{\max}| = \max\{|X| \colon X \text{ is critical}\},$$

and let $v_{\max} \in V_i \setminus X_{\max}$ be a vertex satisfying (C1) for $X_{\max}$, i.e., $v_{\max}$ satisfies

(2.7) $$|\delta^-(X_{\max}; D^* \setminus B)| = f(R(v_{\max})) - 1.$$

From (1.1) and (F1),

(2.8) $$\lambda(v_{\max}, s^*; D^* \setminus B) = f(R(v_{\max})) - 1$$

The following lemma concerning $X_{\max}$ and $v_{\max}$ plays a crucial role in our proof.

Lemma 2.3. *Letting $X_{\max}$ and $v_{\max}$ be those defined above, $f(R(w)) = f(R(v_{\max}))$ holds for every $w \in V_i \setminus (X_{\max} \cup W)$.*

Since the proof of Lemma 2.3 is long, we prove the theorem by using this lemma before giving the proof of Lemma 2.3. The proof of this lemma is given in Section 2.2.

First we prove the following lemma.

Lemma 2.4. *There exists an arc $e = xy$ with $x \in V_i \setminus (X_{\max} \cup W)$ and $y \in W \setminus X_{\max}$ in $D^* \setminus B$.*

Proof. Since a tail and a head of every $e \in B$ are contained in W,

(2.9) $$\delta^-(X_{\max} \cup W; D^* \setminus B) = \delta^-(X_{\max} \cup W; D^*).$$

Next we prove

(2.10) $$|\delta^-(X_{\max} \cup W; D^*)| \geq f(R(v_{\max})).$$

From (C0), there exists $w \in V_i \setminus (X_{\max} \cup W)$. From (1.2), $w \notin X_{\max} \cup W$ and (2.3),

$$|\delta^-(X_{\max} \cup W; D^*)| \underbrace{\geq \lambda(w, s^*; D^*)}_{\text{from (1.2)}} \underbrace{\geq f(R(w))}_{\text{from (2.3)}}.$$

Thus, (2.10) follows from Lemma 2.3. Hence, from (2.7), (2.9) and (2.10)

$$|\delta^-(X_{\max} \cup W; D^* \setminus B)| > |\delta^-(X_{\max}; D^* \setminus B)|.$$

From this inequality, we can see that there exists at least one arc $e = xy$ with $x \in V^* \setminus (X_{\max} \cup W)$ and $y \in W \setminus X_{\max}$. Hence, the lemma holds since $x \in V_i \setminus W$ follows from (2.4) and $y \in W \setminus X_{\max}$. □

Let an arc satisfying Lemma 2.4 be $\hat{e} = \hat{x}\hat{y}$ with $\hat{x} \in V_i \setminus (X_{\max} \cup W)$ and $\hat{y} \in W \setminus X_{\max}$ (see Figure 8). In order to prove that $\hat{e}$ is eligible, from Lemma 2.2, we will prove that there exists no critical set Y such that $\hat{e} \in \delta^-(Y; D^* \setminus B)$.

Lemma 2.5. *There exists no critical set $Y \subseteq V^*$ such that $\hat{e} \in \delta^-(Y; D^* \setminus B)$.*

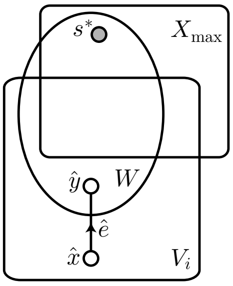

Figure 8: Illustration of $\hat{e}$.

Proof. We will show that if there exists such Y, $X_{\max} \cup Y$ is critical and $\hat{x}$ satisfies (C1) for $X_{\max} \cup Y$. This implies that $|X_{\max}| < |X_{\max} \cup Y|$ holds since $\hat{y} \in Y \setminus X_{\max}$ follows from $\hat{y} \in W \setminus X_{\max}$ and $\hat{y} \in Y$, which contradicts the maximality of $X_{\max}$ in (2.6).

From $\hat{e} \in \delta^-(Y; D^* \setminus B)$, $\hat{x} \notin Y$ holds. Thus, $\hat{x} \in V_i \setminus (X_{\max} \cup Y \cup W)$ holds since $\hat{x} \in V_i \setminus (X_{\max} \cup W)$ follows from the definition of $\hat{e}$. Thus $X_{\max} \cup Y$ satisfies (C0) for $X = X_{\max} \cup Y$.

What remains is to prove that $|\delta^-(X_{\max} \cup Y; D^* \setminus B)| = f(R(\hat{x})) - 1$, i.e., (C1) holds. From $\hat{x} \notin X_{\max} \cup Y$, (1.2) and (F1),

$$|\delta^-(X_{\max} \cup Y; D^* \setminus B)| \underbrace{\geq \lambda(\hat{x}, s^*; D^* \setminus B)}_{\text{from (1.2)}} \underbrace{\geq f(R(\hat{x})) - 1}_{\text{from (F1)}}.$$

Thus, to prove that (C1) holds, it is sufficient to show

$$|\delta^-(X_{\max} \cup Y; D^* \setminus B)| \leq f(R(\hat{x})) - 1. \tag{2.11}$$

Since Y is critical, there exists $w_{\rm cr} \in V_i \setminus Y$ satisfying (C1) for Y, i.e,

$$|\delta^-(Y; D^* \setminus B)| = f(R(w_{\rm cr})) - 1. \tag{2.12}$$

Then, from the submodularity of $|\delta^-(\cdot)|$,

$$\begin{aligned} & f(R(v_{\max})) - 1 + f(R(w_{\rm cr})) - 1 \\ & \underbrace{= |\delta^-(X_{\max}; D^* \setminus B)| + |\delta^-(Y; D^* \setminus B)|}_{\text{by (2.7) and (2.12)}} \\ & \geq |\delta^-(X_{\max} \cap Y; D^* \setminus B)| \\ & \qquad + |\delta^-(X_{\max} \cup Y; D^* \setminus B)|. \end{aligned} \tag{2.13}$$

Since $v_{\max}, w_{\rm cr} \notin X_{\max} \cap Y$ follows from $v_{\max} \notin X_{\max}$ and $w_{\rm cr} \notin Y$, we have

$$\begin{aligned} & |\delta^-(X_{\max} \cap Y; D^* \setminus B)| \\ & \underbrace{\geq \max\{\lambda(v_{\max}, s^*; D^* \setminus B), \lambda(w_{\rm cr}, s^*; D^* \setminus B)\}}_{\text{from (1.2)}} \\ & \underbrace{\geq \max\{f(R(v_{\max})), f(R(w_{\rm cr}))\} - 1}_{\text{from (F1)}}. \end{aligned} \tag{2.14}$$

In the case of $f(R(w_{\rm cr})) \geq f(R(v_{\max}))$, from (2.13) and (2.14), we straightforwardly have

$$|\delta^-(X_{\max} \cup Y; D^* \setminus B)| \leq f(R(v_{\max})) - 1. \tag{2.15}$$

In the case of $f(R(w_{\rm cr})) < f(R(v_{\max}))$, we have $|\delta^-(X_{\max} \cup Y; D^* \setminus B)| \leq f(R(w_{\rm cr})) - 1$ from (2.13) and (2.14), and hence (2.15) follows from $f(R(w_{\rm cr})) < f(R(v_{\max}))$.

Since $\hat{x} \in V_i \setminus (X_{\max} \cup W)$ from the definition of $\hat{e}$, $f(R(\hat{x})) = f(R(v_{\max}))$ follows from Lemma 2.3. Thus, (2.11) follows from (2.15). This completes the proof. □

Proof. [Theorem 1.2] It is not difficult to see that "only if-part" holds. We then prove "if-part". The proof is done by induction on $f(S)$. In the case of $f(S) = 1$, the theorem clearly holds from $|S| = 1$.

Assuming that there exists a feasible in-tree $T = (W, B)$ such that $|W| = l \geq 2$ and $|W| < |V_i|$, we will prove that there exists a feasible in-tree $T' = (W', B')$ such that $|W'| = l + 1$, i.e., there exists an eligible arc for T. If there exists no critical set, it follows from Lemma 2.2 that any $e = uv \in \delta^-(W \setminus \{s^*\}, D^* \setminus B)$ is eligible. In the case where there exists a critical set, letting $X_{\max}$ be a critical set satisfying (2.6), we can see from Lemmas 2.4 and 2.5 that there exists an eligible arc $e = xy$ with $x \in V_i \setminus (X_{\max} \cup W)$ and $y \in W \setminus X_{\max}$. Hence, repeating this process, we eventually have a feasible in-tree $T = (W, B)$ which spans V_i. This completes the proof. □

2.2 Proof of Lemma 2.3

In this section, we prove Lemma 2.3. Lemma 2.3 can be proved by the following two lemmas.

From the definition of a feasible in-tree (F1), $\lambda(w, s^*; D^* \setminus B)$ is at least $f(R(w)) - 1$ for every $w \in V_i$. However, we can see from the following lemma that in fact, $\lambda(w, s^*; D^* \setminus B)$ is equal to $f(R(w)) - 1$ for every $w \in V_i$.

Lemma 2.6. *Letting $T = (W, B)$ be a feasible in-tree, $\lambda(w, s^*; D^* \setminus B) = f(R(w)) - 1$ for every $w \in V_i$.*

Proof. From the way of construction of D^* and the definition of $R(\cdot)$, every set of $f(R(w))$ arc-disjoint paths from w to s^* in D^* use all arcs in $\delta(R(w), \{s^*\}; D^*)$. From (F0), $|\delta(R(w), \{s^*\}; D^* \setminus B)| = f(R(w)) - 1$ follows. Thus, $\lambda(w, s^*; D^* \setminus B) \leq f(R(w)) - 1$ holds since $\lambda(w, s^*; D^* \setminus B)$ is equal to the maximum number of arc-disjoint paths from w to s^* in $D^* \setminus B$. The lemma follows from (F1). □

LEMMA 2.7. *Let $X_{\max}$ be a critical set satisfying (2.6) and $v_{\max}$ be a vertex satisfying (C1) for $X_{\max}$. Then, for every $w \in V_i \setminus (X_{\max} \cup W)$, $\lambda(w, s^*; D^* \setminus B) = \lambda(v_{\max}, s^*; D^* \setminus B)$.*

Proof. We first prove that for every $w \in V_i \setminus (X_{\max} \cup W)$

$$\lambda(w, s^*; D^* \setminus B) \leq \lambda(v_{\max}, s^*; D^* \setminus B). \tag{2.16}$$

Since $w \notin X_{\max}$ follows from $w \in V_i \setminus (X_{\max} \cup W)$,

$$f(R(v_{\max})) - 1 \underbrace{= |\delta^-(X_{\max}; D^* \setminus B)|}_{\text{from (2.7)}} \underbrace{\geq \lambda(w, s^*; D^* \setminus B)}_{\text{from (1.2)}}. \tag{2.17}$$

This inequality and (2.8) imply (2.16).

To prove the lemma, we next show that if there exists $w \in V_i \setminus (X_{\max} \cup W)$ such that

$$\lambda(w, s^*; D^* \setminus B) < \lambda(v_{\max}, s^*; D^* \setminus B), \tag{2.18}$$

there exists a critical set $X \supsetneq X_{\max}$, which contradicts the maximality of $X_{\max}$. Let us fix w as a vertex satisfying (2.18) and

$$U = \{u \in V^* \setminus X_{\max} \colon R(u) \subseteq R(w)\}, \tag{2.19}$$
$$P = X_{\max} \cup (V^* \setminus (X_{\max} \cup U)). \tag{2.20}$$

Notice that $w \in U$ follows from (2.19). Then, we will prove

(U0) $v_{\max} \in P$,

(U1) $|\delta^-(P; D^* \setminus B)| = f(R(w)) - 1$.

(U0) implies $|X_{\max}| < |P|$ since $v_{\max} \notin X_{\max}$ follows from the definition of $v_{\max}$ such that $v_{\max} \in V_i \setminus X_{\max}$ and $X_{\max} \subseteq P$ follows from (2.20). (U1) implies that P is critical from the following two reasons:

- $s^* \in P$ follows from $s^* \in X_{\max}$ and (2.20).
- $w \in V_i \setminus (P \cup W)$ holds since (i) $w \notin W$ follows from $w \in V_i \setminus (X_{\max} \cup W)$, and (ii) $w \notin P$ follows from $w \notin X_{\max}$ and $w \in U$ and (2.20).

This contradicts the maximality of $X_{\max}$ in (2.6).

Now let us prove (U0). It is sufficient to prove $v_{\max} \notin U$ since $v_{\max} \notin U$ implies $v_{\max} \in V^* \setminus (X_{\max} \cup U)$ from $v_{\max} \notin X_{\max}$, and hence $v_{\max} \in P$ follows from (2.20). To prove $v_{\max} \notin U$, we will show $R(v_{\max}) \not\subseteq R(w)$ since this implies $v_{\max} \notin U$ from (2.19). Assuming $R(v_{\max}) \subseteq R(w)$, from the definition of $f(\cdot)$,

$$f(R(v_{\max})) \leq f(R(w)). \tag{2.21}$$

From (F1), (2.21) and (2.8),

$$\begin{aligned} &\lambda(w, s^*; D^* \setminus B) \\ &\quad \geq f(R(w)) - 1 \quad \text{(from (F1))} \\ &\quad \geq f(R(v_{\max})) - 1 \quad \text{(from (2.21))} \\ &\quad = \lambda(v_{\max}, s^*; D^* \setminus B) \quad \text{(from (2.8))}. \end{aligned}$$

This contradicts (2.18). Thus, (U0) holds.

Next we prove (U1). We first show

$$\delta^-(P; D^* \setminus B) \subseteq \delta^-(X_{\max}; D^* \setminus B). \tag{2.22}$$

To prove (2.22), from (2.20), it is sufficient to prove

$$\delta(U \setminus X_{\max}, V^* \setminus (X_{\max} \cup U); D^* \setminus B) = \emptyset, \tag{2.23}$$

since from $(U \setminus X_{\max}) \cup (V^* \setminus (X_{\max} \cup U)) = V^* \setminus X_{\max}$, (2.23) implies (2.22) (see Figure 9). Assuming that there exists an arc $e = xy$ in the arc set of the left hand side of (2.23), $R(y) \subseteq R(x)$ follows from the definition of $R(\cdot)$. From (2.19), $x \in U$ implies $R(x) \subseteq R(w)$, and also $y \notin U$ implies $R(y) \not\subseteq R(w)$. This contradicts $R(y) \subseteq R(x)$. Thus, (2.23) holds.

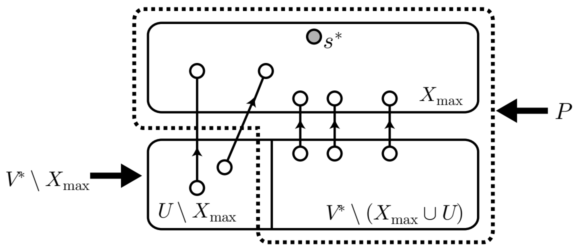

Figure 9: Illustration of (2.23)

From now on, we prove (U1) by using (2.22). Recalling that $w \notin P$ follows from $w \notin X_{\max}$, $w \in U$ and (2.20),

$$|\delta^-(P; D^* \setminus B)| \underbrace{\geq \lambda(w, s^*; D^* \setminus B)}_{\text{from (1.2)}} \underbrace{\geq f(R(w)) - 1}_{\text{from (F1)}}.$$

Thus, to prove (U1), it is sufficient to prove that $|\delta^-(P; D^* \setminus B)| \leq f(R(w)) - 1$. Assuming $|\delta^-(P; D^* \setminus$

$B)| > f(R(w)) - 1$,

$$|\delta^-(X_{\max}; D^* \setminus B) \setminus \delta^-(P; D^* \setminus B)|$$
$$\underbrace{= |\delta^-(X_{\max}; D^* \setminus B)| - |\delta^-(P; D^* \setminus B)|}_{\text{by (2.22)}}$$
$$\underbrace{= (f(R(v_{\max})) - 1) - |\delta^-(P; D^* \setminus B)|}_{\text{by (2.7)}}$$
$$\underbrace{< (f(R(v_{\max})) - 1) - (f(R(w)) - 1)}_{\text{by the assumption made above}}$$
$$= f(R(v_{\max})) - f(R(w)). \tag{2.24}$$

However, as will be shown below, (2.24) contradicts that there exist $f(R(v_{\max})) - 1$ arc-disjoint paths from $v_{\max}$ to s^* in $D^* \setminus B$. To prove this statement, we prove

(S0) there exist at most $f(R(w)) - 1$ arc-disjoint paths from $v_{\max}$ to s^* in $D^* \setminus B$ which use arcs in $\delta^-(P; D^* \setminus B)$.

(S0) implies that there exist less than $f(R(v_{\max})) - 1$ arc-disjoint paths from $v_{\max}$ to s^* since

- the number of arc-disjoint paths from $v_{\max}$ to s^* in $D^* \setminus B$ that use arcs in $\delta^-(X_{\max}; D^* \setminus B) \setminus \delta^-(P; D^* \setminus B)$ is less than $f(R(v_{\max})) - f(R(w))$ from (2.24), and

- the number of arc-disjoint paths from $v_{\max}$ to s^* in $D^* \setminus B$ that use arcs in $\delta^-(P; D^* \setminus B)$ is at most $f(R(w)) - 1$ from (S0).

Here we prove (S0). Let H be the set of heads of all arcs in $\delta^-(P; D^* \setminus B)$. If we can prove $(\bigcup_{h \in H} R(h)) \subseteq R(w)$, (S0) holds since $|\delta(R(w), \{s^*\}; D^* \setminus B)| = f(R(w)) - 1$ follows from the definition of D^* and (F0). Assume that there exist $s_j \in (\bigcup_{h \in H} R(h)) \setminus R(w)$ and $e = xy \in \delta^-(P; D^* \setminus B)$ such that $s_j \in R(y)$. Notice that $e \in \delta^-(P; D^* \setminus B)$ implies $x \notin P$, and hence $x \notin P$ implies $x \in U$ from (2.20). Thus, since $R(x) \subseteq R(w)$ follows from $x \in U$ and $R(y) \subseteq R(x)$ follows from the definition of $R(\cdot)$, $s_j \in R(y)$ implies $s_j \in R(w)$. This contradicts $s_j \in (\bigcup_{h \in H} R(h)) \setminus R(w)$, and completes the proof of (S0). Hence (U1) is proved by (S0), and this completes the proof of the lemma. □

Proof. [Lemma 2.3] For every $w \in V_i \setminus (X_{\max} \cup W)$,

$$f(R(w)) - 1 \underbrace{= \lambda(w, s^*; D^* \setminus B)}_{\text{from Lemma 2.6}} \underbrace{= \lambda(v_{\max}, s^*; D^* \setminus B)}_{\text{from Lemma 2.7}}.$$

The lemma follows from this equality and (2.8). □

3 Another Characterization of Packing In-trees

Edmonds [1] showed the following another characterization of packing in-trees. For directed graphs $D_1 = (W_1, B_1)$ and $D_2 = (W_2, B_2)$, the union of D_1 and D_2 is defined as $D_3 = (W_1 \cup W_2, B_1 \cup B_2)$. We call a subgraph T of D *tree* when T has no cycle in the graph obtained by ignoring the direction of arcs of D. Here we define a *feasible set of k trees* $\mathcal{T}$ in D with specified vertex $s \in V$ as a set of k arc-disjoint trees such that each tree spans V and for every $v \in V$

$$|\delta^+(\{v\}; F)| = \begin{cases} k, & \text{if } v \in V \setminus \{s\}, \\ 0, & \text{if } v = s, \end{cases}$$

where F is the union of k arc-disjoint trees in $\mathcal{T}$.

THEOREM 3.1. ([1]) *Given a directed graph $D = (V, A)$ with a specified vertex $s \in V$, there exists a feasible set of k trees if and only if $\lambda(v, s; D) \geq k$ holds for every $v \in V \setminus \{s\}$.*

We extend this characterization to the one in our case as follows. Here we define a *feasible set of $f(S)$ subtrees* $\mathcal{T}^*$ in D^* as a set of $f(S)$ arc-disjoint trees denoted by $T^*_{i,1}, T^*_{i,2}, \ldots, T^*_{i,f(s_i)}$ for every $i = 1, \ldots, d$ such that each $T^*_{i,j}$ spans $V_i \cup \{s^*\}$ and for every $v \in V$

$$|\delta^+(\{v\}; F^*)| = \begin{cases} f(R(v)), & \text{if } v \in V, \\ 0, & \text{if } v = s^*, \end{cases}$$

where F^* is the union of $f(S)$ trees in $\mathcal{T}^*$.

The proof of the following theorem is based on the proof of Theorem 3.1 of Gabow (see Corollary 2.1 in [3]).

THEOREM 3.2. *Given a directed graph $D = (V, A)$ with a set of specified vertices $S = \{s_1, \ldots, s_d\}$ and a function $f: S \to \mathbb{N}$, there exists a feasible set of $f(S)$ subtrees if and only if $\lambda(v, s^*; D^*) \geq f(R(v))$ holds for every $v \in V$.*

Proof. **If-part**: Since $T^*_{i,1}, T^*_{i,2}, \ldots, T^*_{i,f(s_i)}$ for every $i = 1, \ldots, d$ which compose a feasible set of $f(S)$ subtrees $\mathcal{T}^*$ can be straightforwardly constructed from $T_{i,1}, T_{i,2}, \ldots, T_{i,f(s_i)}$ for every $i = 1, \ldots, d$ in Theorem 1.2, it is clear to see that "if-part" follow from Theorem 1.2.
Only if-part: Suppose that there exist $T^*_{i,j}$s which compose a feasible set of $f(S)$ subtrees $\mathcal{T}^*$. From (1.1), the statement that $\lambda(v, s^*; D^*) \geq f(R(v))$ holds for every $v \in V$ is equivalent to

(i) $\delta^+(W; D^*) \geq f(R(v))$ holds for every $v \in V$ and $W \subseteq V$ with $v \in W$.

Thus, we will prove the statement (i). Let us fix $v \in V$ and $W \subseteq V$ with $v \in W$. Recall that F^* is the union of $f(S)$ trees in $\mathcal{T}^*$. Thus, precisely $\sum_{w \in W} f(R(w))$ arcs of F^* have their tails in W from the definition of $\mathcal{T}^*$ and $s^* \notin W$. Here let I_W be the set of $i \in \{1, \ldots, d\}$ such that $W \cap V_i \neq \emptyset$. Here we consider the sum of the number of arcs of $T^*_{i,1}, T^*_{i,2}, \ldots, T^*_{i,f(s_i)}$ which have both ends in W for $i \in I_W$. Since $T^*_{i,j}$ is a tree and spans V_i, the number of arcs of $T^*_{i,j}$ which have both ends in W is equal to $|W \cap V_i| - 1$. Thus, $\sum_{i \in I_W}(|W \cap V_i| - 1) \cdot f(s_i)$ arcs of F^* have both ends in W. Thus, to prove the statement (i), since

$$\delta^+(W; D^*) \geq \sum_{w \in W} f(R(w)) - \sum_{i \in I_W} (|W \cap V_i| - 1) \cdot f(s_i)$$

follows from the above discussion, it is sufficient to prove

$$\sum_{w \in W} f(R(w)) - \sum_{i \in I_W} (|W \cap V_i| - 1) \cdot f(s_i) \geq f(R(v)). \tag{3.25}$$

Recalling that $f(R(w)) = \sum_{s_i \in R(w)} f(s_i)$ holds,

$$\sum_{w \in W} f(R(w)) = \sum_{w \in W} \sum_{s_i \in R(w)} f(s_i). \tag{3.26}$$

Since $s_i \in R(w)$ implies $w \in V_i$,

$$\begin{aligned} \sum_{w \in W} \sum_{s_i \in R(w)} f(s_i) &= \sum_{i \in \{1,\ldots,d\}} \sum_{w \in W \cap V_i} f(s_i) \\ &= \underbrace{\sum_{i \in I_W} \sum_{w \in W \cap V_i} f(s_i)}_{\text{from } W \cap V_i = \emptyset \text{ for } i \notin I_W} \\ &= \sum_{i \in I_W} |W \cap V_i| \cdot f(s_i). \end{aligned} \tag{3.27}$$

Thus, the left hand side of (3.25) is equal to $\sum_{i \in I_W} f(s_i)$ from (3.26) and (3.27). Hence (3.25) follows from $R(v) \subseteq \{s_i : i \in I_W\}$ for every $v \in W$. □

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